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Journal of Functional Analysis 239 (2006) 76–99
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Ginzburg–Landau minimizers with prescribed degrees. Capacity of the domain and emergence of vortices

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Received 30 November 2005; accepted 1 March 2006

Available online 15 May 2006

Communicated by H. Brezis

Abstract

Let $\Omega \subset \mathbf{R}^2$ be a simply connected domain, let ω be a simply connected subdomain of Ω, and set $A = \Omega \setminus \omega$. Suppose that $\mathcal{J}$ is the class of complex-valued maps on the annular domain A with degree 1 both on $\partial\Omega$ and on $\partial\omega$. We consider the variational problem for the Ginzburg–Landau energy E_λ among all maps in $\mathcal{J}$. Because only the degree of the map is prescribed on the boundary, the set $\mathcal{J}$ is not necessarily closed under a weak H^1-convergence. We show that the attainability of the minimum of E_λ over $\mathcal{J}$ is determined by the value of $\operatorname{cap}(A)$—the H^1-capacity of the domain A. In contrast, it is known, that the existence of minimizers of E_λ among the maps with a prescribed Dirichlet boundary data does not depend on this geometric characteristic. When $\operatorname{cap}(A) \geqslant \pi$ (A is either *subcritical* or *critical*), we show that the global minimizers of E_λ exist for each $\lambda > 0$ and they are vortexless when λ is large. Assuming that $\lambda \to \infty$, we demonstrate that the minimizers of E_λ converge in $H^1(A)$ to an S^1-valued harmonic map which we explicitly identify. When $\operatorname{cap}(A) < \pi$ (A is *supercritical*), we prove that either (i) there is a critical value λ_0 such that the global minimizers exist when $\lambda < \lambda_0$ and they do not exist when $\lambda > \lambda_0$, or (ii) the global minimizers exist for each $\lambda > 0$. We conjecture that the second case never occurs. Further, for large λ, we establish that the minimizing sequences/minimizers in supercritical domains develop exactly two vortices—a vortex of degree 1 near $\partial\Omega$ and a vortex of degree -1 near $\partial\omega$.

Keywords: Ginzburg–Landau vortices; Elliptic PDE's with limited compactness

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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2006.03.006

1. Introduction

Consider the following problem:

$$m_\lambda := \operatorname{Inf}\left\{E_\lambda(u) = \frac{1}{2}\int_A |\nabla u|^2 + \frac{\lambda}{4}\int_A \left(1-|u|^2\right)^2;\ u \in \mathcal{J}\right\}. \tag{1.1}$$

Here, λ is a nonnegative real number, E_λ is a Ginzburg–Landau (GL) energy functional, A is a two-dimensional annular domain, i.e. $A = \Omega \setminus \omega$, $\overline{\omega} \subset \Omega$, where Ω and ω are simply connected, bounded smooth domains. The class $\mathcal{J}$ of testing maps is

$$\mathcal{J} = \left\{u \in H^1\left(A; \mathbb{R}^2\right);\ |u| = 1 \text{ a.e. on } \partial A,\ \deg(u, \partial\Omega) = \deg(u, \partial\omega) = 1\right\}. \tag{1.2}$$

The definition of $\mathcal{J}$ is meaningful. Indeed, let Γ be $\partial\Omega$ or $\partial\omega$ (with the counterclockwise orientation) and set $X = H^{1/2}(\Gamma; S^1)$. If $u \in H^1(A; \mathbb{R}^2)$ and $|u| = 1$ a.e. on ∂A, then $g := u|_\Gamma \in X$ (here, the restriction is to be understood in the sense of traces). Maps in X have a well-defined topological degree (winding number), see [12]. This degree is defined as follows: every map $g \in X$ is the strong $H^{1/2}$-limit of a sequence $(g_n) \subset C^\infty(\Gamma; S^1)$. Each g_n has a degree (with respect to the counterclockwise orientation on Γ) given, e.g. by the classical formula

$$\deg g_n = \frac{1}{2\pi}\int_\Gamma g_n \times g_{n,\tau}. \tag{1.3}$$

Then $\lim_n \deg g_n$ exists [19] and the degree of the map g can be defined as $\deg g = \lim_n \deg g_n$. Note that the formula (1.3) is still valid for arbitrary maps in X, provided we interpret the integral via an $H^{1/2}$–$H^{-1/2}$ duality.

We now address a natural question concerning the minimization problem (1.1)–(1.2).

Question 1. Is m_λ attained?

We start by recalling the most extensively studied minimization problem for the GL functional:

$$e_\lambda := \operatorname{Inf}\left\{E_\lambda(u);\ u|_{\partial G} = g\right\}, \tag{1.4}$$

see [10]. Here, G is a smooth bounded domain in $\mathbb{R}^2$ and $g \in H^{1/2}(\partial G; S^1)$ is fixed. In this case, e_λ is obviously attained, since the class $\{u \in H^1(G); u|_{\partial G} = g\}$ is closed with respect to weak-H^1 convergence.

The situation is more delicate when, instead of the Dirichlet boundary condition, only a degree of a map is prescribed on the boundary as shown by the following example.

Example 1. (Inf is not attained [7].) Let

$$n_\lambda := \operatorname{Inf}\left\{E_\lambda(u);\ u \in \mathcal{M}\right\}, \tag{1.5}$$

where

$$\mathcal{M} = \left\{u \in H^1(\mathbb{D}_1);\ |u| = 1 \text{ a.e. on } S^1,\ \deg\left(u, S^1\right) = 1\right\}, \tag{1.6}$$

$\mathbb{D}_1$ is the unit disk and we consider the counterclockwise orientation on S^1. Then, for each $\lambda > 0$, $n_\lambda = \pi$ and n_λ is not attained.

In particular, this example implies that the class $\mathcal{M}$ is not closed with respect to weak-H^1 convergence. It is possible to construct an explicit example of a sequence in $\mathcal{M}$ weakly converging in H^1 to a map which is not in $\mathcal{M}$.

Example 2. [8] Let $(a_n) \subset (0,1)$ be such that $a_n \to 1$. Set

$$u_n(z) = \frac{z - a_n}{1 - a_n z}, \quad z \in \mathbb{D}_1.$$

Then $u_n \rightharpoonup -1$ weakly in H^1.

Example 2 can be easily extended to $\mathcal{J}$:

Proposition 1. [7] *The class $\mathcal{J}$ is not closed with respect to weak-H^1 convergence.*

The immediate consequence of this proposition is that the existence of minimizers of (1.1)–(1.2) cannot be established by using the direct method of calculus of variations.

Before discussing Question 1 further, we mention some useful a priori bounds on m_λ. Recall that, in the case of a prescribed Dirichlet data with non-zero degree [10], the GL energy tends to infinity as $\lambda \to \infty$. However, a straightforward calculation shows that the energy remains bounded (with a bound independent of A and λ) when only the degrees of the boundary data are prescribed:

$$m_\lambda \leqslant 2\pi, \tag{1.7}$$

see [8].

There is yet another upper bound, which is obtained by considering all S^1-valued maps in $\mathcal{J}$. Set

$$\mathcal{K} = \{u \in \mathcal{J};\ |u| = 1 \text{ a.e. in } A\}. \tag{1.8}$$

$\mathcal{K}$ is not empty: if $a \in \omega$, then $(x-a)/|x-a| \in \mathcal{K}$. It is known that the minimum of E_λ is attained in $\mathcal{K}$ [10]. Define

$$I_0 = \operatorname{Min}\{E_\lambda(u);\ u \in \mathcal{K}\} = \operatorname{Min}\left\{\frac{1}{2}\int_A |\nabla u|^2;\ u \in \mathcal{K}\right\}. \tag{1.9}$$

Proposition 2. *We have*

$$m_\lambda < I_0. \tag{1.10}$$

Clearly, (1.7) and (1.10) imply that $m_\lambda \leqslant \operatorname{Min}\{I_0, 2\pi\}$. This bound is almost optimal when λ is large:

Proposition 3. *The asymptotic behavior of m_λ is given by the equality*

$$\lim_{\lambda\to\infty} m_\lambda = \mathrm{Min}\{I_0, 2\pi\}. \tag{1.11}$$

It turns out that I_0 has a simple geometrical interpretation via capacity:

Proposition 4. [7] *I_0 and the H^1-capacity,* $\mathrm{cap}(A)$*, of the domain A are related by*

$$I_0 = \frac{2\pi^2}{\mathrm{cap}(A)}. \tag{1.12}$$

Recall that the H^1-capacity is given by

$$\mathrm{cap}(A) = \mathrm{Min}\left\{\int_A |\nabla u|^2;\ u \in H^1(A),\ u|_{\partial\Omega} = 0,\ u|_{\partial\omega} = 1\right\}.$$

For example, if $A = \{x;\ r < |x| < R\}$, then $\mathrm{cap}(A) = 2\pi/\ln(R/r)$. In general, $\mathrm{cap}(A)$ is a measure of the "thickness" of A.

Formula (1.11) and the relationship (1.12) between $\mathrm{cap}(A)$ and I_0 suggest that there are three different types of domains:

(a) "subcritical" or "thin," when $\mathrm{cap}(A) > \pi$ (or, equivalently, $I_0 < 2\pi$);
(b) "critical," when $\mathrm{cap}(A) = \pi$ (or, equivalently, $I_0 = 2\pi$);
(c) "supercritical" or "thick," when $\mathrm{cap}(A) < \pi$ (or, equivalently, $I_0 > 2\pi$).

We now return to the question of existence of minimizers. The main tool in proving the existence is the following.

Proposition 5. *Assume that $m_\lambda < 2\pi$. Then m_λ is attained.*

The results of this type were first established for the Yamabe problem by T. Aubin in [5] and subsequently proved to be extremely useful in minimization problems with possible lack of compactness of minimizing sequences; see [13,16,17,19] and the more recent papers [18,22,28]. The proof of Proposition 5 relies on the following lemma.

Lemma 1 *(Price lemma). Let (u_n) be a bounded sequence in $\mathcal{J}$ such that $u_n \rightharpoonup u$ in $H^1(A)$. Then*

$$\liminf_n \frac{1}{2}\int_A |\nabla u_n|^2 \geqslant \frac{1}{2}\int_A |\nabla u|^2 + \pi\big(\big|1-\deg(u,\partial\Omega)\big| + \big|1-\deg(u,\partial\omega_0)\big|\big). \tag{1.13}$$

In addition,

$$\frac{1}{2}\int_A |\nabla u|^2 \geqslant \pi\big|\deg(u,\partial\Omega) - \deg(u,\partial\omega)\big|. \tag{1.14}$$

The argument we use here works for arbitrary fixed degrees [7]. The general form of the estimate (1.13) shows [7] that the minimal energy needed to jump from the degree d (for the maps u_n) to the degree δ (for the map u) on a given connected component of ∂A is $\pi|d-\delta|$.

As an immediate consequence of Proposition 5 and the upper bound (1.10), we obtain the following theorem.

Theorem 1. *Assume that A is subcritical or critical. Then m_λ is attained for each $\lambda \geqslant 0$.*

In the subcritical and critical case, we further address the following natural question.

Question 2. What is the behavior of minimizers u_λ of (1.1)–(1.2) as $\lambda \to \infty$?

The answer is given by the following theorem.

Theorem 2. *Let* $\mathrm{cap}(A) \geqslant \pi$, *i.e. A is subcritical or critical. Let u_λ be a minimizer of* (1.1)–(1.2). *Then $|u_\lambda| \to 1$ uniformly in $\overline{A}$ as $\lambda \to \infty$. In addition, up to a subsequence, $u_\lambda \to u_\infty$ in $H^1(A)$, where u_∞ is a minimizer of* (1.8)–(1.9).

Theorem 2 combined with the method developed in [9] yield the stronger convergence $u_\lambda \to u_\infty \in C^{1,\alpha}(\overline{A})$, $0<\alpha<1$; see [7]. We also prove in [7] that, for large λ, minimizers are unique modulo multiplication by a constant in S^1 and are symmetric if the domain itself is symmetric. Whenever minimizers u_λ exist, they are smooth [7]. This is not a standard regularity result, because the boundary conditions satisfied by the u_λ's are of mixed type—Dirichlet for the modulus $|u_\lambda|$ and Neumann for the phase $\arg u_\lambda$.

We now turn to the supercritical case $\mathrm{cap}(A) < \pi$. Concerning existence of minimizers, we prove that there are exactly two possibilities (see Fig. 1).

Theorem 3. *Let* $\mathrm{cap}(A) < \pi$, *that is let A be supercritical. Then either*:

(a) *m_λ is attained for all λ ($m_\lambda < 2\pi$); or*
(b) *there exists a critical value $\lambda_1 \in (0,\infty)$ such that m_λ is attained when $\lambda < \lambda_1$ while it is not attained when $\lambda > \lambda_1$.*

In contrast with supercritical/critical case, we prove that minimizing sequences (or minimizers, if they exist) must develop vortices (zeros of non-zero degree). If A is a circular annulus, the

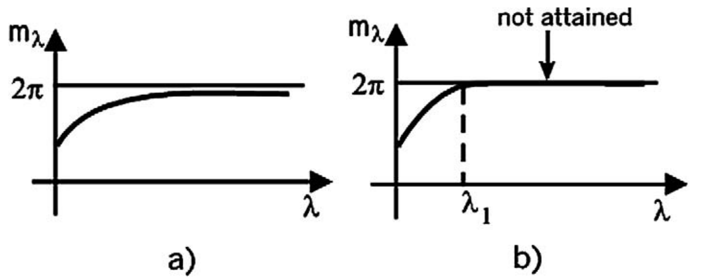

Fig. 1. m_λ vs. λ.

presence of vortices indicates the “breaking of symmetry” of the minimizer—similar phenomenon in problems for harmonic maps was studied in [1,11]. The inheritance of symmetry of the domain by minimizers of the GL functional was considered in [27].

Theorem 4 *(Rise of vortices). Let A be supercritical.*

In case (a) *of Theorem* 3, *let u_λ be a minimizer of* (1.1)–(1.2). *Then, for large λ, the map u_λ has exactly two simple zeros ζ_λ and ξ_λ of degrees* 1 *and* -1, *respectively, such that $\zeta_\lambda \to \partial\Omega$ and $\xi_\lambda \to \partial\omega$ as $\lambda \to \infty$.*

In case (b) *of Theorem* 3, *fix $\lambda > \lambda_1$ and let (u_λ^k) be a minimizing sequence for* (1.1)–(1.2). *Then $u_\lambda^k = v_\lambda^k + w_\lambda^k$, where $w_\lambda^k \to 0$ in $H^1(A)$ as $k \to \infty$ and v_λ^k is smooth. Further, the map v_λ^k satisfies the GL equation and has exactly two simple zeros, ζ_k and ξ_k, of degrees* 1 *and* -1, *respectively, such that $\zeta_k \to \partial\Omega$ and $\xi_k \to \partial\omega$ as $k \to \infty$.*

We introduce the decomposition $u_\lambda^k = v_\lambda^k + w_\lambda^k$ because u_λ^k belongs merely to H^1 and thus need not be continuous. Although there is no natural notion of zeros of u_λ^k, it is meaningful to consider zeros of v_λ^k, because it is a smooth map. An intuitive interpretation of this statement is that u_λ^k *essentially* has two zeros for large k.

Further, in case (b), we prove (Step 5 in the proof of Theorem 4 in Section 4) that, near ζ_k, the map u_λ^k essentially behaves as a conformal map Φ_k from Ω into $\mathbb{D}_1$ with $\Phi_k(\zeta_k) = 0$ and, near ξ_k, as an anti-conformal map $\overline{\Psi_k}$ from $\mathbb{C} \setminus \overline{\omega}$ into $\mathbb{D}_1$ with $\overline{\Psi_k}(\xi_k) = 0$. A similar conclusion is valid in case (a) as well.

We believe that case (a) *never* occurs, hence we propose the following

Conjecture. *In the supercritical case, there exists a* finite *value $\lambda_1 > 0$ such that m_λ is never attained when $\lambda > \lambda_1$.*

A formal argument in support of this conjecture is as follows. Assume that case (a) holds. For a large λ, let $d = \operatorname{dist}(\{\zeta_\lambda, \xi_\lambda\}, \partial A)$ (cf. Theorem 4). It is easy to verify that

$$\frac{\lambda}{4}\int_A \left(1 - |u_\lambda|^2\right)^2 \geqslant C_1 \lambda d^2.$$

On the other hand, various examples suggest that

$$\frac{1}{2}\int_A |\nabla u_\lambda|^2 \geqslant 2\pi - C_2 d^2,$$

where C_1 and C_2 do not depend on λ or d. If it can be proved that the second inequality does indeed hold, then the upper bound (1.7) contradicts the existence of minimizers for large λ.

Finally, we discuss specific features of the critical case. It is known that, for variational problems with lack of compactness, the critical case could inherit the properties of either the supercritical or the subcritical case [11,15,18,20]. In our problem, the results are the same in critical and subcritical case, the supercritical case being qualitatively different. However, while the proof of the existence is the same in both subcritical and critical cases, the proof of H^1-convergence of the minimizers u_λ as $\lambda \to \infty$ for the subcritical case cannot be extended to the critical case and a more subtle argument is required.

We conclude the introduction with a brief review of existing work on minimization of GL functionals related to the problem considered in this paper.

The GL functionals have been extensively studied for general domains. The asymptotics as $\lambda \to \infty$ of global minimizers for the GL functional and their vortex structure for the Dirichlet boundary data (for which the degree is fixed) was considered by Bethuel et al. in [9,10]. The existence and the qualitative behavior of minimizers in [9,10] do not depend on the size (capacity) of the domain.

A minimization problem for the GL functional with the magnetic field in simply connected domains for classes of functions with no prescribed boundary conditions was studied by Serfaty [31,32] and by Sandier and Serfaty [30]. In this case, the qualitative changes in the behavior of minimizers are described in terms of a parameter defined by the external magnetic field. In particular, the existence of a threshold value for this parameter corresponding to a transition from vortex-less minimizers to minimizers with vortices was proved in [31] . For multiply connected domains, a similar result for Bose–Einstein condensate was established by Aftalion et al. [2]. More recently Alama and Bronsard [4] considered energy minimizers for the GL functional with magnetic field in multiply connected domains, in the London limit as the GL parameter tends to infinity. Note that in the present work we establish an existence result for an *arbitrary* value of the GL parameter in the absence of the magnetic field.

The existence of local minimizers for the GL functional with the magnetic field over three-dimensional tori was considered by Rubinstein and Sternberg [29]. Their approach relies on the fact that, when the GL parameter λ is large, the boundedness of the nonlinear term in the GL energy forces the minimizing maps to be, in some sense, "close" to S^1-valued maps. The first step in their proof consists of finding, for $\lambda = \infty$, local minimizers for the GL functional in different homotopy classes of S^1-valued maps (existence of these homotopy classes is due to White [33]). This step is reminiscent of the method used by Brezis and Coron in [17] for harmonic maps. The next step consists of proving, for λ large, the existence of local minimizers close to the ones obtained for $\lambda = \infty$. These existence results are not influenced by the domain size (capacity). Note that [29] generalizes the earlier results of Jimbo and Morita [25] obtained for solids of revolution with a convex cross-section.

If adapted to our case, the methods of [29] yield, for large λ, the existence of local minimizers in $\mathcal{J}$ that are H^1-close to the minimizers of E_λ in $\mathcal{K}$. If A is subcritical or supercritical, it can be proved that these critical points are, for large λ, the genuine minimizers [7]. However, they are not minimizers when λ is large and A is supercritical.

The minimization problem for GL functional with the degree boundary conditions in a special case of a narrow circular annulus was studied by Golovaty and Berlyand [24]. The techniques developed there rely on the radial symmetry and cannot be applied to general domains.

Finally we mention that some of the results presented in this paper were announced in [6].

2. Existence of minimizers

The following simple fact will be repeatedly used in the sequel. Let (u_n) be a bounded sequence in $H^1(A)$ such that for every n we have $|u_n| = 1$ a.e. on ∂A. If $u_n \rightharpoonup u$ in H^1, clearly $|u| = 1$ a.e. on ∂A and both $\deg(u, \partial\Omega)$ and $\deg(u, \partial\omega)$ are well defined.

Proof of the Price lemma. Set $v_n = u_n - u$. We have

$$\int_A |\nabla u_n|^2 = \int_A |\nabla u|^2 + \int_A |\nabla v_n|^2 + o(1), \tag{2.1}$$

as $n \to \infty$. Given an arbitrary $f \in C^\infty(\overline{A}; [-1, 1])$, we integrate by parts the pointwise inequality $|\nabla v_n|^2 \geqslant 2f \operatorname{Jac} v_n$ to obtain

$$\int_A |\nabla v_n|^2 \geqslant 2\int_A f \operatorname{Jac} v_n = \int_{\partial A} f v_n \times v_{n,\tau} + \int_A (f_x v_{n,y} \times v_n - f_y v_{n,x} \times v_n), \tag{2.2}$$

where ∂A has the counterclockwise orientation. The above equality follows from the identity

$$2 \operatorname{Jac} v_n = (v_n \times v_{n,y})_x + (v_{n,x} \times v_n)_y,$$

when v_n is smooth. The same inequality for $v_n \in H^1$ follows by approximation. Since $v_n \rightharpoonup 0$ in H^1, (2.1) and (2.2) yield

$$\int_A |\nabla u_n|^2 \geqslant \int_A |\nabla u|^2 + \int_{\partial A} f v_n \times v_{n,\tau} + o(1), \tag{2.3}$$

via an embedding argument. On the other hand, if Γ is any connected component of ∂A, then

$$\int_\Gamma v_n \times v_{n,\tau} = \int_\Gamma u_n \times u_{n,\tau} - \int_\Gamma u \times u_\tau + o(1). \tag{2.4}$$

Indeed, if $g_n \rightharpoonup g$ in $H^{1/2}(\Gamma)$ and $h \in H^{1/2}(\Gamma)$, then

$$\int_\Gamma g_n h_\tau = \int_\Gamma g \times h_\tau + o(1) \quad \text{and} \quad \int_\Gamma h \times g_{n,\tau} = \int_\Gamma h \times g_\tau + o(1), \tag{2.5}$$

where the integrals are understood in the sense of an $H^{1/2}$–$H^{-1/2}$ duality. Then (2.4) follows easily from (2.5) and the fact that $u_n|_\Gamma \rightharpoonup u|_\Gamma$ in $H^{1/2}(\Gamma)$.

Now choose f such that $f = \operatorname{sgn}(1 - \deg(u, \partial\Omega))$ on $\partial\Omega$, $f = -\operatorname{sgn}(1 - \deg(u, \partial\omega))$ on $\partial\omega$, and $-1 \leqslant f \leqslant 1$ in A. By combining (2.1), (2.3), (2.4), and the degree formula (1.3), we obtain (1.13).

Equation (1.14) relies on the pointwise inequality $|\nabla u|^2 \geqslant 2|\operatorname{Jac} u|$, which yields

$$\int_A |\nabla u|^2 \geqslant 2\int_A |\operatorname{Jac} u| \geqslant 2\left|\int_A \operatorname{Jac} u\right| = \left|\int_{\partial A} u \times u_\tau\right| = 2\pi\left|\deg(u, \partial\Omega) - \deg(u, \partial\omega)\right|, \tag{2.6}$$

following an integration by parts and taking into account (1.3). □

Proof of Proposition 5. Let (u_n) be a minimizing sequence for E_λ in $\mathcal{J}$. Up to a subsequence, we can assume that $u_n \rightharpoonup u$ for some $u \in H^1(A)$. Set $D = \deg(u, \partial\Omega)$ and $d = \deg(u, \partial\omega)$. If $d = D = 1$, then $u \in \mathcal{J}$ and u is a minimizer of (1.1)–(1.2). If both $D \neq 1$ and $d \neq 1$ then (1.13) implies that

$$2\pi > m_\lambda = \lim_n E_\lambda(u_n) \geqslant \liminf_n \frac{1}{2}\int_A |\nabla u_n|^2 \geqslant \pi\big(|1 - d| + |1 - D|\big) \geqslant 2\pi, \tag{2.7}$$

which is a contradiction. Finally, if only one of two integers d and D is equal to 1, then $|d-D| \geqslant 1$ and $|1-d|+|1-D| \geqslant 1$. By combining (1.13) and (1.14) we obtain $m_\lambda \geqslant 2\pi$ once again—this is impossible. $\square$

Proof of Proposition 2. Let u be a minimizer of (1.8)–(1.9) and set $g = u|_{\partial A}$. If v minimizes E_λ among all the maps $w \in H^1(A)$ such that $w|_{\partial A} = g$, then $v \in \mathcal{J}$ and $m_\lambda \leqslant E_\lambda(v) \leqslant E_\lambda(u) = I_0$. We claim that the last inequality is strict. Arguing by contradiction, assume that $E_\lambda(v) = E_\lambda(u)$. Then u minimizes E_λ with respect to its own boundary conditions; in particular, u satisfies the GL equation $-\Delta u = \lambda u(1-|u|^2)$. Since $|u| = 1$ a.e., we find that u is harmonic and has the modulus 1. Thus u has to be a constant, which contradicts the fact that $u \in \mathcal{K}$. $\square$

Proof of Theorem 3. The mapping $\lambda \mapsto m_\lambda$ is clearly both non-decreasing and continuous. In view of the upper bound (1.7), there is some $\lambda_1 \in [0, \infty]$ such that $m_\lambda < 2\pi$ if $\lambda < \lambda_1$, and $m_\lambda = 2\pi$ if $\lambda \geqslant \lambda_1$. We first claim that m_λ is not attained if $\lambda > \lambda_1$. Arguing by contradiction, we assume that there are some $\lambda > \lambda_1$ and $u \in \mathcal{J}$ such that $E_\lambda(u) = m_\lambda = 2\pi$. As in the proof of Proposition 2, we cannot have $|u| = 1$ a.e. Thus

$$\int_A \left(1-|u|^2\right)^2 > 0$$

and, therefore, $E_{\lambda'}(u) < E_\lambda(u)$ if $\lambda' < \lambda$. For any λ' such that $\lambda_1 < \lambda' < \lambda$, this implies that $m_{\lambda'} \leqslant E_{\lambda'}(u) < 2\pi$, which is a contradiction. $\square$

In view of Proposition 2, m_λ is attained for all $\lambda < \lambda_1$. Case (a) in Theorem 3 corresponds to $\lambda_1 \in (0, \infty)$ and case (b) to $\lambda_1 = \infty$. Therefore, in order to complete the proof of Theorem 3, it remains to rule out the possibility that $\lambda_1 = 0$. This amounts to proving the following lemma.

Lemma 2. *We have $m_0 < 2\pi$.*

Proof. We start by considering a circular annulus, $A = \{z \in \mathbb{R}^2;\ r < |z| < R\}$. Set

$$u(z) = \frac{z}{R+r} + \frac{rR}{(R+r)\bar{z}}.$$

It is easy to check that $u(z) = z/|z|$ on ∂A, so that $u \in \mathcal{J}$. On the other hand, it is also straightforward to verify that

$$E_0(u) = 2\pi \frac{R-r}{R+r} < 2\pi,$$

hence $m_0 < 2\pi$.

Consider now the case of a general A. Recall that there is a conformal representation Φ of A into some circular annulus $\mathcal{A}$; moreover, Φ extends to a C^1-diffeomorphism of $\overline{A}$ into $\overline{\mathcal{A}}$ and we may choose Φ in order to preserve the orientation of curves [3]. Let $F : H^1(\mathcal{A}) \to H^1(A)$, $F(u) = u \circ \Phi$. If $\mathcal{J}(A)$ and $\mathcal{J}(\mathcal{A})$ stand for the corresponding classes of testing maps, we claim that F is a bijection between $\mathcal{J}(\mathcal{A})$ and $\mathcal{J}(A)$. In order to prove this statement, we have to show that the degrees on the connected components of the boundary are preserved by Φ. Indeed, let Γ be a connected component of ∂A and let $\gamma = \Phi(\Gamma)$. Since Φ is orientation-preserving, we have

$$\deg(g, \gamma) = \deg(g \circ \Phi, \Gamma) \tag{2.8}$$

for $g \in C^\infty(\gamma; S^1)$. Using the density of $C^\infty(\gamma; S^1)$ in $H^{1/2}(\gamma; S^1)$ and the continuity of the map $g \mapsto g \circ \Phi$ from $H^{1/2}(\gamma; S^1)$ into $H^{1/2}(\Gamma; S^1)$, we find that (2.8) is still valid for $g \in H^{1/2}(\gamma; S^1)$. Thus F maps $\mathcal{J}(\mathcal{A})$ into $\mathcal{J}(A)$. Similarly, F^{-1} maps $\mathcal{J}(A)$ into $\mathcal{J}(\mathcal{A})$. So F is a bijection between $\mathcal{J}(\mathcal{A})$ and $\mathcal{J}(A)$.

Using the conformal invariance of the Dirichlet integral, we find that m_0 has the same value for both A and $\mathcal{A}$. Since $m_0 < 2\pi$ for circular annuli, the proof of Lemma 2 is complete. □

3. Proof of Theorem 2

Let u_λ be a minimizer of (1.1)–(1.2) for a given $\lambda \geqslant 0$. We start by observing that a sequence (u_λ) is bounded in $H^1(A)$. Indeed, the upper bound (1.7) implies that (∇u_λ) is bounded in $L^2(A)$. Thus, by a Poincaré-type inequality, $u_\lambda - a_\lambda$ is bounded in $H^1(A)$, where $a_\lambda = (1/|\partial\Omega|)\int_{\partial\Omega} u_\lambda$. Since $|u_\lambda| = 1$ a.e. on $\partial\Omega$, a_λ is bounded, so that u_λ is bounded in $H^1(A)$.

Let $u_\infty \in H^1(A)$ be such that, up to some subsequence, $u_{\lambda_n} \rightharpoonup u_\infty$ in $H^1(A)$. In view of (1.7), we have

$$\int_A \left(1 - |u_\lambda|^2\right)^2 \to 0,$$

and thus $u_\infty \in H^1(A; S^1)$.

In the subcritical case, we identify u_∞ using the Price lemma and the following simple lemma.

Lemma 3. *Let $u \in H^1(A; S^1)$. Then* $\deg(u, \partial\Omega) = \deg(u, \partial\omega)$.

Proof. Differentiating the equality $|u|^2 = 1$ a.e. we find that $u \cdot u_x = u \cdot u_y = 0$ a.e., so that $\operatorname{Jac} u = 0$ a.e. On the other hand, an integration by parts used in conjunction with the degree formula (1.3) yields

$$0 = \int_A \operatorname{Jac} u = \frac{1}{2}\int_{\partial A} u \times u_\tau = \pi\big(\deg(u, \partial\Omega) - \deg(u, \partial\omega)\big). \qquad \square \tag{3.1}$$

For the convenience, we divide the remainder of the proof of Theorem 2 into five steps.

Step 1. Identification of u_∞ and strong-$H^1(A)$ convergence in the subcritical case.

By combining the Price lemma, Proposition 2, Lemma 3, and the upper bound (1.10), we have

$$2\pi > I_0 \geqslant \liminf_n m_{\lambda_n} \geqslant \liminf_n \frac{1}{2}\int_A |\nabla u_{\lambda_n}|^2 \geqslant \frac{1}{2}\int_A |\nabla u_\infty|^2 + 2\pi\big|1 - \deg(u_\infty, \partial\Omega)\big|, \tag{3.2}$$

in the subcritical case $I_0 < 2\pi$. It follows from (3.2) that $\deg(u_\infty, \partial\Omega) = \deg(u_\infty, \partial\omega) = 1$, that is $u_\infty \in \mathcal{K}$, and $I_0 \geqslant \frac{1}{2}\int_A |\nabla u_\infty|^2$. Recalling the definition of I_0, we find that u_∞ minimizes (1.8)–(1.9). Then it follows from (3.2) that

$$I_0 \geqslant \liminf_n \frac{1}{2}\int_A |\nabla u_{\lambda_n}|^2 \geqslant \frac{1}{2}\int_A |\nabla u_\infty|^2 = I_0, \tag{3.3}$$

which implies that $u_{\lambda_n} \to u_\infty$ in $H^1(A)$.

Step 2. An improved upper bound for m_λ.

The following result is a slight improvement of the upper bound (1.10).

Lemma 4. *There exist constants $C > 0$ and $\lambda_0 > 0$, such that $m_\lambda \leqslant I_0 - \frac{C}{\lambda}$ for $\lambda > \lambda_0$.*

Proof. Let u minimize (1.8)–(1.9), then $u \in C^\infty(\overline{A})$ [10]. Consider an arbitrary $f \in C_0^\infty(A; \mathbb{R})$ and set $v_\lambda = (1 - f/\lambda)u$. The map v_λ coincides with u on ∂A and thus belongs to $\mathcal{J}$. It is easy to observe that $|\nabla v_\lambda|^2 = (1 - f/\lambda)^2|\nabla u|^2 + |\nabla f|^2/\lambda^2$, since u is S^1-valued. Thus

$$m_\lambda \leqslant E_\lambda(v_\lambda) = \frac{1}{2}\int_A |\nabla u|^2 - \frac{1}{\lambda}\int_A f\left(|\nabla u|^2 - f\right) + O\left(\frac{1}{\lambda^2}\right). \tag{3.4}$$

The conclusion of Lemma 4 follows from (3.4) by choosing f such that $0 \leqslant f \leqslant |\nabla u|^2$ in A and $0 < f < |\nabla u|^2$ in some nonempty open subset of A. □

Step 3. Candidates for u_∞ in the critical case.

Lemma 5. *Assume that A is critical. Then either u_∞ minimizes* (1.8)–(1.9)*, or u_∞ is identically equal to a constant of modulus* 1.

Proof. We rely on the Price lemma, Lemma 3, and the upper bound (1.7). As in (3.2), we have

$$2\pi = I_0 \geqslant \liminf_n m_{\lambda_n} \geqslant \frac{1}{2}\int_A |\nabla u_\infty|^2 + 2\pi\left|1 - \deg(u_\infty, \partial\Omega)\right|. \tag{3.5}$$

If $\deg(u_\infty, \partial\Omega) = \deg(u_\infty, \partial\omega) = 1$ then, as in Step 1, we find that u_∞ minimizes (1.8)–(1.9). On the other hand, if $\deg(u_\infty, \partial\Omega) = \deg(u_\infty, \partial\omega) \neq 1$, then (3.5) implies that u_∞ must be identically equal to a constant. Since $|u_\infty| = 1$ a.e. on ∂A, this constant is of modulus 1. □

Step 4. Identification of u_∞ and strong-$H^1(A)$ convergence in the critical case.

We rely on the following lemma.

Lemma 6. [26] *Let* (v_λ) *be a family of solutions of the GL equation* $-\Delta v_\lambda = \lambda v_\lambda(1-|v_\lambda|^2)$ *in* A. *Assume that* $|v_\lambda| \leqslant 1$ *and* $E_\lambda(v_\lambda) \leqslant C$ *uniformly in* λ. *Then* (v_λ) *is bounded in* $C^\infty_{\mathrm{loc}}(A)$. *In addition, the following pointwise estimates hold*:

$$1 - \left|v_\lambda(z)\right|^2 \leqslant \frac{D}{\lambda d^2(z)}, \quad z \in A, \tag{3.6}$$

and

$$\left|D^k v_\lambda(z)\right| \leqslant \frac{D_k}{d^k(z)}, \quad z \in A,\ k \in \mathbb{N}; \tag{3.7}$$

here, $d(z) = \operatorname{dist}(z, \partial A)$ *and the constants* D, D_k *depend only on* C.

In order to identify u_∞, we rule out the possibility that u_∞ is a constant. We argue by contradiction. Let Γ be a simple curve in A enclosing $\partial\omega$. Let U be the domain enclosed by $\partial\Omega$ and Γ and set $V = A \setminus \overline{U}$. Integrating the pointwise inequality $|\nabla u_\lambda|^2 \geqslant 2\operatorname{Jac} u_\lambda$ over U and using the degree formula (1.3), we find that

$$\frac{1}{2}\int_U |\nabla u_\lambda|^2 \geqslant \pi - \frac{1}{2}\int_\Gamma u_\lambda \times u_{\lambda,\tau}, \tag{3.8}$$

where Γ is counterclockwise oriented. Similarly, the inequality $|\nabla u_\lambda|^2 \geqslant -2\operatorname{Jac} u_\lambda$ yields

$$\frac{1}{2}\int_V |\nabla u_\lambda|^2 \geqslant \pi - \frac{1}{2}\int_\Gamma u_\lambda \times u_{\lambda,\tau}. \tag{3.9}$$

Thus

$$m_\lambda \geqslant \frac{1}{2}\int_A |\nabla u_\lambda|^2 \geqslant 2\pi - \int_\Gamma u_\lambda \times u_{\lambda,\tau}. \tag{3.10}$$

Next we observe that u_λ satisfies the assumption of Lemma 6 for every λ. Indeed, any minimizer of (1.1)–(1.2) satisfies the GL equation. Since $|u_\lambda| = 1$ a.e. on ∂A, we have $|u_\lambda| \leqslant 1$ in A, by the maximum principle [9]. Finally, we have $E_\lambda(u_\lambda) \leqslant 2\pi$ for each λ.

Since u_∞ is a constant, in view of Lemma 6, we have for large λ that $1/2 \leqslant |u_\lambda| \leqslant 1$ on Γ and $\deg(u_\lambda, \Gamma) = 0$. Thus u_λ admits the representation $u_\lambda = \rho_\lambda e^{\imath\varphi_\lambda}$ on Γ for large λ. Here $1/2 \leqslant \rho_\lambda \leqslant 1$ and φ_λ is *single-valued*. Therefore, we have

$$\int_\Gamma u_\lambda \times u_{\lambda,\tau} = \int_\Gamma \rho_\lambda^2 \varphi_{\lambda,\tau} = \int_\Gamma \left(\rho_\lambda^2 - 1\right)\varphi_{\lambda,\tau}. \tag{3.11}$$

On the other hand, the assumption that u_∞ is a constant and Lemma 6 imply that $\nabla\varphi_\lambda \to 0$ uniformly on Γ, as $\lambda \to \infty$. This fact, in conjunction with (3.11) and the estimate (3.6), yield

$$\int_\Gamma u_\lambda \times u_{\lambda,\tau} = o\left(\frac{1}{\lambda}\right). \tag{3.12}$$

Equation (3.12) along with (3.10) imply that

$$m_\lambda \geqslant 2\pi - o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty. \tag{3.13}$$

Since inequality (3.13) contradicts the conclusion of Lemma 4, for large λ, it follows that u_∞ is not a constant. In view of Step 3, the map u_∞ minimizes (1.8)–(1.9), hence $u_{\lambda_n} \to u_\infty$ strongly in H^1 (cf. Step 1).

Step 5. Show that $|u_\lambda| \to 1$ uniformly in $\overline{A}$ as $\lambda \to \infty$.

As we have, the sequence (u_λ) is bounded in $H^1(A)$. Moreover, if $u_{\lambda_n} \to u_\infty$ weakly in H^1, it follows from Steps 1 and 4 that $u_{\lambda_n} \to u_\infty$ strongly in H^1 and u_∞ minimizes (1.8)–(1.9). For such a sequence (u_{λ_n}), it remains to prove that $|u_{\lambda_n}| \to 1$ uniformly in $\overline{A}$ as $n \to \infty$.

Fix some $a \in (0,1)$. We have to establish the inequality

$$\left|u_{\lambda_n}(z)\right| \geqslant a \quad \text{in } A \text{ for large } n. \tag{3.14}$$

Recall the following lemma.

Lemma 7. [20] *Let $g_n, g \in \mathrm{VMO}(\partial A; S^1)$ be such that $g_n \to g$ in* VMO. *Let $\tilde{g}_n$, $\tilde{g}$ be the corresponding harmonic extensions to A. Then, for each $\varepsilon > 0$, there is some $\delta = \delta(\varepsilon) > 0$ (independent of n) such that*

$$\left|\tilde{g}_n(z)\right| \geqslant 1 - \varepsilon \quad \text{if } d(z) \leqslant \delta. \tag{3.15}$$

Lemma 8. [9] *Let $v \in H_0^1(A)$ be such that $\Delta v \in L^\infty$. Then, for some C depending only on A, we have*

$$\|\nabla v\|_{L^\infty} \leqslant C\|v\|_{L^\infty}^{1/2}\|\Delta v\|_{L^\infty}^{1/2}. \tag{3.16}$$

Set $g_n = u_{\lambda_n}|_{\partial A}$, $g = u_\infty|_{\partial A}$. Since $H^{1/2}(\partial A) \subset \mathrm{VMO}(\partial A)$ and $u_{\lambda_n} \to u_\infty$ in $H^1(A)$, we find that $g_n \to g$ in VMO. We consider a decomposition $u_{\lambda_n} = \tilde{g}_n + v_{\lambda_n}$, where $v_{\lambda_n} \in H_0^1(A)$ is the solution of $-\Delta v_{\lambda_n} = \lambda_n u_{\lambda_n}(1 - |u_{\lambda_n}|^2)$. Observe that

$$|v_{\lambda_n}| \leqslant |\tilde{g}_n| + |u_{\lambda_n}| \leqslant 2. \tag{3.17}$$

Here we rely on the inequality $|u_{\lambda_n}| \leqslant 1$ and on the fact that, $\tilde{g}_n$ being the harmonic extension of a map of modulus 1, itself has the modulus that does not exceed 1. Using Lemma 8 in conjunction with (3.17) and the definition of v_{λ_n}, we find that

$$|\nabla v_{\lambda_n}| \leqslant C\sqrt{2\lambda_n}. \tag{3.18}$$

Then

$$\left|v_{\lambda_n}(z)\right| \leqslant C_1\sqrt{\lambda_n}\, d(z) \tag{3.19}$$

for some C_1 independent of n, since $v_{\lambda_n}=0$ on ∂A. Combining (3.19) with Lemma 7 we obtain that there exist constants $C_2=C_2(a)$ and $n_0=n_0(a)$, such that

$$\left|u_{\lambda_n}(z)\right|\geqslant a \quad \text{if } d(z)\leqslant\frac{C_2}{\sqrt{\lambda_n}} \text{ and } n\geqslant n_0. \tag{3.20}$$

Returning to the proof of (3.14), we proceed as in [9]. We argue by contradiction. Suppose that (up to a subsequence) there are points $z_n\in A$ such that $|u_{\lambda_n}(z_n)|\leqslant a$. In view of (3.20), we have

$$d(z_n)\geqslant\frac{C_2}{\sqrt{\lambda_n}}, \tag{3.21}$$

for large n.

By (3.7), given an arbitrary $C_3\in(0,C_2)$, there exists a constant $C_4>0$ independent of n and such that $|\nabla u_{\lambda_n}(z)|\leqslant C_4\sqrt{\lambda_n}$ when $|z-z_n|\leqslant C_3/\sqrt{\lambda_n}$. Since $|u_{\lambda_n}(z_n)|\leqslant a$, we thus have

$$\left|u_{\lambda_n}(z)\right|\leqslant\frac{1+a}{2} \quad \text{if } |z-z_n|\leqslant\frac{C_3}{\sqrt{\lambda_n}} \text{ and } n \text{ is large}, \tag{3.22}$$

provided we choose C_3 sufficiently small. For such C_3 and for a sufficiently large n, we have

$$\lambda_n\int\limits_A\left(1-|u_{\lambda_n}|^2\right)^2\geqslant\lambda_n\int\limits_{\Pi_n}\left(1-|u_{\lambda_n}|^2\right)^2\geqslant C_5, \tag{3.23}$$

where $\Pi_n:=\{z;\ |z-z_n|\leqslant C_3/\sqrt{\lambda_n}\}$ and C_5 is independent of n.

On the other hand, the upper bound (1.10), the strong-H^1 convergence $u_{\lambda_n}\to u_\infty$, together with the fact that u_∞ minimizes (1.8)–(1.9) yield

$$I_0\geqslant\lim_n\left(\frac12\int\limits_A|\nabla u_{\lambda_n}|^2+\frac{\lambda_n}{4}\int\limits_A\left(1-|u_{\lambda_n}|^2\right)^2\right)=I_0+\lim_n\frac{\lambda_n}{4}\int\limits_A\left(1-|u_{\lambda_n}|^2\right)^2. \tag{3.24}$$

Thus we must have

$$\lim_n\frac{\lambda_n}{4}\int\limits_A\left(1-|u_{\lambda_n}|^2\right)^2=0. \tag{3.25}$$

For large n, Eqs. (3.23) and (3.25) contradict each other. Therefore, (3.14) holds and the proof of Theorem 2 is complete.

4. Rise of vortices

Except when otherwise noted, we assume that the domain A is supercritical throughout this section. First suppose that case (a) in Theorem 3 holds. As noted at the beginning of the proof of Theorem 2, the family (u_λ) is bounded in $H^1(A)$. Thus, up to a subsequence, $u_{\lambda_n}\rightharpoonup u_\infty$, where $u_\infty\in H^1(A;S^1)$. Next suppose that case (b) in Theorem 3 holds. Consider a minimizing sequence (u^k) for a fixed $\lambda>\lambda_1$. By the same argument as above, (u^k) is bounded in $H^1(A)$ and, up to a subsequence, $u^{k_n}\rightharpoonup u_\infty$, where $u_\infty\in H^1(A;\mathbb{C})$. We begin by identifying u_∞.

Lemma 9. *In both cases in Theorem* 3 *the map* u_∞ *is identically equal to a constant of modulus* 1.

Proof. Assume first case (a). By combining the Price lemma, the upper bound (1.7), and Lemma 3, we find that

$$2\pi \geqslant \liminf_n \frac{1}{2}\int_A |\nabla u_{\lambda_n}|^2 \geqslant \frac{1}{2}\int_A |\nabla u_\infty|^2 + 2\pi\left|1-\deg(u_\infty,\partial\Omega)\right|. \tag{4.1}$$

If $\deg(u_\infty,\partial\Omega)\neq 1$, then $\nabla u_\infty = 0$ a.e. and u_∞ has to be a constant. This constant is of modulus 1, since $|u_\infty| = 1$ a.e. on ∂A. On the other hand, if $\deg(u_\infty,\partial\Omega) = 1$, then $u_\infty \in \mathcal{K}$ and (4.1) yields

$$2\pi \geqslant \frac{1}{2}\int_A |\nabla u_\infty|^2 \geqslant I_0, \tag{4.2}$$

but in the supercritical case $I_0 > 2\pi$. Thus u_∞ is a constant of modulus 1.

Next assume case (b). As the proof of Theorem 3 shows, $m_\lambda = 2\pi$ for $\lambda > \lambda_1$. The Price lemma implies that

$$2\pi = m_\lambda = \lim_n E_\lambda\left(u^{k_n}\right) \geqslant E_\lambda(u_\infty) + \pi\left(\left|1-\deg(u_\infty,\partial\Omega)\right| + \left|1-\deg(u_\infty,\partial\omega)\right|\right). \tag{4.3}$$

If $\deg(u_\infty,\partial\Omega) = \deg(u_\infty,\partial\omega) = 1$, then $u_\infty \in \mathcal{J}$ and u_∞ minimizes (1.1)–(1.2) by (4.3). This, however, is impossible, since m_λ is not attained for $\lambda > \lambda_1$. If $\deg(u_\infty,\partial\Omega)\neq 1$ and $\deg(u_\infty,\partial\omega)\neq 1$, then (4.3) implies that u_∞ has to be equal to a constant (of modulus 1). Finally, if exactly one among $\deg(u_\infty,\partial\Omega)$ and $\deg(u_\infty,\partial\omega)$ equals 1, then (4.3) combined with (1.14) yields

$$2\pi \geqslant 2\pi + \frac{\lambda}{4}\int_A \left(1-|u_\infty|^2\right)^2. \tag{4.4}$$

Therefore, u_∞ is a constant of modulus 1, which is in contradiction with the assumption on the degrees of u_∞. We conclude that u_∞ is a constant of modulus 1. □

As a byproduct of the above lemma, it is easy to establish Proposition 3.

Proof of Proposition 3. Since m_λ is not decreasing, for each sequence $\lambda_n \to \infty$ we have $\lim_{\lambda\to\infty} m_\lambda = \lim_n m_{\lambda_n}$.

Assume first that A is subcritical or critical. Consider a sequence (λ_n) such that $u_{\lambda_n} \to u_\infty$ strongly in $H^1(A)$, where u_∞ minimizes (1.8)–(1.9). By combining the upper bound (1.10) with the definition of I_0, we find that

$$I_0 \geqslant \lim_{\lambda\to\infty} m_\lambda = \lim_n E_{\lambda_n}(u_{\lambda_n}) \geqslant \frac{1}{2}\int_A |\nabla u_\infty|^2 = I_0. \tag{4.5}$$

Thus $\lim_{\lambda\to\infty} m_\lambda = I_0$, as claimed.

Assume next that A is supercritical. In case (b), we have $m_\lambda = 2\pi$ for large λ and, thus, (1.11) holds. In case (a), consider a sequence (λ_n) such that $u_{\lambda_n} \rightharpoonup u_\infty$ weakly in $H^1(A)$, where u_∞ is a constant of modulus 1. Using the Price lemma and the upper bound (1.7), we obtain

$$2\pi \geqslant \lim_{\lambda\to\infty} m_\lambda = \lim_n E_{\lambda_n}(u_{\lambda_n}) \geqslant 2\pi, \tag{4.6}$$

which yields $\lim_{\lambda\to\infty} m_\lambda = 2\pi$ and (1.11) follows. □

Proof of Theorem 4: case (b). For $\lambda > \lambda_1$, we consider the behavior of a minimizing sequence (u^k). For the convenience of the reader, we divide the proof into six steps.

Step 1. Decomposition of u^k.

Suppose that v^k minimizes the GL energy E_λ among all maps $v \in H^1(A)$ such that $v = u^k$ on ∂A. Clearly,

(i) v^k satisfies the GL equation $-\Delta v^k = \lambda v^k(1 - |v^k|^2)$,
(ii) $|v^k| \leqslant 1$ (by the maximum principle),
(iii) $v^k \in \mathcal{J}$, and
(iv) the sequence (v^k) is still a minimizing sequence for E_λ in $\mathcal{J}$, since $E_\lambda(v^k) \leqslant E_\lambda(u^k)$.

Set $w^k = u^k - v^k \in H^1_0(A)$.

Lemma 10. *We have $w^k \to 0$ in $H^1(A)$ as $k \to \infty$.*

Proof. In view of Lemma 9, we may assume that, up to a subsequence, $u^k_n \rightharpoonup u$ and $v^k_n \rightharpoonup v$ weakly in $H^1(A)$, where u, v are constants of modulus 1. Since $u^k = v^k$ on ∂A we have $u = v$ and, hence, $w^k_n \rightharpoonup 0$. In fact, since this conclusion holds for every subsequence of the original sequence, it follows that $w^k \rightharpoonup 0$ weakly in $H^1(A)$.

Inserting the equality $u^k = v^k + w^k$ into the expression for $E_\lambda(u^k)$ and using the fact that $w^k \rightharpoonup 0$, we obtain

$$E_\lambda(u^k) = E_\lambda(v^k) + \frac{1}{2}\int_A |\nabla w^k|^2 + \int_A \nabla v^k \cdot \nabla w^k + o(1). \tag{4.7}$$

Furthermore,

$$\frac{1}{2}\int_A |\nabla w^k|^2 + \int_A \nabla v^k \cdot \nabla w^k \to 0 \quad \text{as } k \to \infty, \tag{4.8}$$

since both (u^k) and (v^k) are minimizing sequences. On the other hand, if we multiply by w^k the GL equation satisfied by v^k and integrate, we find that

$$\left| \int_A \nabla v^k \cdot \nabla w^k \right| = \left| \int_A \lambda v^k \cdot w^k (1 - |v^k|^2) \right| \leqslant \lambda \int_A |w^k| \to 0 \quad \text{as } k \to \infty, \tag{4.9}$$

by an embedding argument. Equation (4.8) when used in conjunction with (4.9) yields

$$\lim_k \int_A |\nabla w^k|^2 = 0.$$

Since $w^k = 0$ on ∂A, we find that $w^k \to 0$ in $H^1(A)$ by the Poincaré's inequality and Lemma 10 follows. □

In conclusion, modulo a small remainder w_k in $H^1(A)$, we may replace a minimizing sequence (u^k) by the minimizing sequence (v^k), having two additional properties (i) and (ii). In the rest of the proof, we will study the behavior of the sequence (v^k).

Step 2. Concentration of the energy near ∂A.

We fix two simple curves γ and Γ in A, such that γ encloses $\partial\omega$ and Γ encloses γ. Let U be the domain enclosed by $\partial\Omega$ and Γ, V be the domain enclosed by γ and $\partial\omega$ and set $W = A \setminus (\overline{U} \cup \overline{V})$.

Lemma 11. *When $k \to \infty$, we have*

$$\int_A (1 - |v^k|^2)^2 \to 0, \tag{4.10}$$

$$\|\nabla v^k\|_{L^\infty(W)} \to 0, \tag{4.11}$$

$$\|\partial_{\bar z} v^k\|_{L^2(U)} \to 0 \quad \text{and} \quad \|\partial_z v^k\|_{L^2(V)} \to 0, \tag{4.12}$$

$$\frac{1}{2}\int_U |\nabla v^k|^2 \to \pi \quad \text{and} \quad \int_U \operatorname{Jac} v^k \to \pi, \tag{4.13}$$

$$\frac{1}{2}\int_V |\nabla v^k|^2 \to \pi \quad \text{and} \quad \int_V \operatorname{Jac} v^k \to -\pi. \tag{4.14}$$

Proof. We integrate the identities

$$\frac{1}{2}|\nabla v^k|^2 = \operatorname{Jac} v^k + 2|\partial_{\bar z} v^k|^2$$

and

$$\frac{1}{2}|\nabla v^k|^2 = -\operatorname{Jac} v^k + 2|\partial_z v^k|^2$$

over U and V, respectively. We find that

$$E_\lambda(v^k) = \int_U \operatorname{Jac} v^k - \int_V \operatorname{Jac} v^k + 2\int_U |\partial_{\bar z} v^k|^2 + 2\int_V |\partial_z v^k|^2 + \frac{1}{2}\int_W |\nabla v^k|^2 + \frac{\lambda}{4}\int_A (1 - |v^k|^2)^2. \tag{4.15}$$

An integration by parts combined with the degree formula (1.3) yields,

$$\int_U \operatorname{Jac} v^k = \pi - \frac{1}{2}\int_\Gamma v^k \times v^k_\tau \quad \text{and} \quad -\int_U \operatorname{Jac} v^k = \pi - \frac{1}{2}\int_\gamma v^k \times v^k_\tau \tag{4.16}$$

for the counterclockwise orientation on γ and Γ.

We claim that

$$\nabla v^k \to 0 \quad \text{in } C^0_{\mathrm{loc}}(A), \tag{4.17}$$

as $k \to \infty$. Then the conclusions of Lemma 11 can be obtained as follows. Using (4.17) we pass to the limit in (4.16) and, in turn, in (4.15). Here we take into account the facts that $|v^k| \leqslant 1$ and $\lim_k E_\lambda(v^k) = 2\pi$.

It remains to establish (4.17). Since $|v^k| \leqslant 1$, we have that $|\Delta v^k| \leqslant \lambda$. Since the sequence (v^k) is bounded in H^1, it follows from standard elliptic estimates [23] that (v^k) is bounded in $W^{2,p}_{\mathrm{loc}}(A)$ for every $1 < p < \infty$. Furthermore, (v^k) is relatively compact in $C^1_{\mathrm{loc}}(A)$ due to the Sobolev embeddings. In view of Lemma 9, each subsequence of (v^k) contains a further subsequence converging weakly in H^1 to a constant map of modulus 1. It is easy to see that this property, along with the fact that (v^k) is relatively compact in $C^1_{\mathrm{loc}}(A)$, implies (4.17). Note for further use, that the same argument implies that $|v^k| \to 1$ in $C^1_{\mathrm{loc}}(A)$. □

Step 3. Existence of zeros.

Lemma 12. *There is some $k_0 \in \mathbf{N}$ such that, for $k \geqslant k_0$, the map v^k has at least one zero ζ_k in U, at least one zero ξ_k in V, and no zeros in $\overline{W}$. In addition, for every zero ζ'_k in U and ξ'_k in V, we have* $\operatorname{dist}(\zeta'_k, \partial\Omega) \to 0$ *and* $\operatorname{dist}(\xi'_k, \partial\omega) \to 0$, *respectively, as $k \to \infty$.*

Proof. Non-existence of zeros in $\overline{W}$ for large λ and the last property follow from the fact that $|v^k| \to 1$ in $C^1_{\mathrm{loc}}(A)$. It remains to establish existence of zeros in U and in V for large λ.

We argue by contradiction. Assume, for example, that, up to a subsequence, $v^k \neq 0$ in U. Then we claim that, for every k, there exists $C_k > 0$ such that $C_k \leqslant |v^k| \leqslant 1$ in $\overline{U}$. Since $|v^k| \to 1$ in $C^1_{\mathrm{loc}}(U)$, it remains to show that v^k is bounded away from zero near $\partial\Omega$. Indeed, Lemma 7 applied to $g = v^k|_{\partial A}$, $g_n \equiv g$, implies that there is some $\delta_1 > 0$ such that $\tilde{g}(z) \geqslant 3/4$ if $d(z) < \delta_1$. On the other hand, if we set $w^k = v^k - \tilde{g}(z) \in H^1_0(A)$, then $\Delta w^k \in L^\infty(A)$ and thus $w^k \in C^1_0(\overline{A})$. Therefore, there is some $\delta_2 > 0$ such that $|w^k(z)| \leqslant 1/4$ if $d(z) < \delta_2$. We conclude that $|v^k(z)| \geqslant 1/2$ if $d(z) < \mathrm{Min}(\delta_1, \delta_2)$ and the claim follows.

Set $y_k = v^k/|v^k|$. This map belongs to $H^1(U; S^1)$, since $C_k \leqslant |v^k| \leqslant 1$ in U. Due to Lemma 3 we have $\deg(y_k, \Gamma) = \deg(y_k, \partial\Omega)$, hence $\deg(y_k, \Gamma) = 1$ since $y_k = v^k$ on $\partial\Omega$. Therefore $\deg(v_k, \Gamma) = \deg(y_k, \Gamma) = 1$. This is impossible since, up to a subsequence, $v^k \to v$ in $C^1(\Gamma)$, and v is a constant of modulus 1. The proof of Lemma 12 is complete. □

Step 4. Rescaling of v^k.

Recall that $\nabla v^k \to 0$ and $|v^k| \to 1$ in $C^1(\Gamma)$. Thus, we can extend $v^k|_U$ to Ω so that the extension v^k_1 satisfies $\|\nabla v^k_1\|_{L^\infty(\Omega\setminus U)} \to 0$ and $1/2 \leqslant |v^k_1| \leqslant 1$ in $\Omega \setminus U$ for large k. Similarly,

$v^k|_V$ has an extension v_2^k to $\mathbb{C} \setminus \overline{\omega}$ satisfying $\|\nabla v_2^k\|_{L^\infty(\mathbb{C}\setminus V)} \to 0$ and $1/2 \leqslant |v_2^k| \leqslant 1$ in $\mathbb{C} \setminus V$ for large k.

Let Φ be a fixed conformal representation of Ω into $\mathbb{D}_1$. It is well known that conformal representations Φ_k of Ω into $\mathbb{D}_1$ satisfying the property $\Phi_k(\zeta_k) = 0$ are given by

$$\Phi_k(z) = \alpha \frac{\Phi(z) - \Phi(\zeta_k)}{1 - \overline{\Phi(\zeta_k)}\Phi(z)}, \quad \text{where } \alpha \in S^1.$$

Set $y_k = v_1^k \circ \Phi_k^{-1}$. By construction, y_k maps $\mathbb{D}_1$ into $\mathbb{D}_1$ and vanishes at the origin; moreover, the trace of y_k on S^1 has modulus 1 and degree 1 (since Φ_k preserves the orientation of curves). It is easy to see that, for an appropriate choice of α, we may assume that $\partial_z y_k(0) \geqslant 0$. Similarly, we may construct a conformal representation Ψ_k of $\mathbb{C} \setminus \overline{\omega}$ onto $\mathbb{D}_1$ vanishing at ξ_k and such that $z_k = \overline{v_2^k \circ \Psi_k^{-1}}$ has the same properties as y_k.

In the remaining part of the proof, we study the asymptotic properties of y_k and z_k and relate these properties to the asymptotic behavior of v^k. The reason we prefer to deal with y_k and z_k instead of v^k is a lack of strong-H^1 convergence: as we have already seen, up to a subsequence, $v^{k_n} \rightharpoonup v$, where v is some constant of modulus 1. In particular, (v^{k_n}) is not strongly convergent in H^1, since the degrees change in the limit. However, as we will establish below, y_k and z_k *do* strongly converge in $H^1(\mathbb{D}_1)$. We focus on the behavior of y_k; the analysis for z_k is the same.

Recall some elementary properties of the Φ_k.

Lemma 13. [7] *For every $r \in (0, 1)$, there are constants $C_j = C_j(r)$ independent of k and such that*:

(i) $\Phi_k^{-1}(\mathbb{D}_r) \subset \{z \in \Omega;\ |z - \zeta_k| \leqslant C_1 d(\zeta_k, \partial\Omega)$ *and* $d(z, \partial\Omega) \geqslant C_2 d(\zeta_k, \partial\Omega)\}$;
(ii) $|\nabla \Phi_k^{-1}| \leqslant C_3 d(\zeta_k, \partial\Omega)$ *in* $\mathbb{D}_r$.

For each $R_1, R_2 > 0$, there is an $r \in (0, 1)$ independent of k such that

(iii) $\Phi_k(\{z \in \Omega;\ |z - \zeta_k| \leqslant R_1 d(\zeta_k, \partial\Omega)$ *and* $d(z, \partial\Omega) \geqslant R_2 d(\zeta_k, \partial\Omega)\}) \subset \mathbb{D}_r$.

Lemma 14. *We have $y_k \to \mathrm{id}$ and $z_k \to \mathrm{id}$ strongly in $H^1(\mathbb{D}_1)$ and in $C^1_{\mathrm{loc}}(\mathbb{D}_1)$.*

Proof. Since the Dirichlet integral is conformally invariant, using Lemma 11 we have

$$\int_{\mathbb{D}_1} |\nabla y_k|^2 = \int_{\Omega} \left|\nabla v_1^k\right|^2 = \int_{U} \left|\nabla v^k\right|^2 + \int_{\Omega\setminus U} \left|\nabla v_1^k\right|^2 = 2\pi + o(1), \tag{4.18}$$

as $k \to \infty$. Similarly

$$\int_{\mathbb{D}_1} \left(|\nabla y_k|^2 - 2\,\mathrm{Jac}\, y_k\right) = o(1), \tag{4.19}$$

as $k \to \infty$.

The fact that $|y_k| \leqslant 1$, combined with (4.18) implies that (y_k) is bounded in $H^1(\mathbb{D}_1)$. Let $y \in H^1(\mathbb{D}_1)$ be such that, up to a subsequence, $y_{k_n} \rightharpoonup y$. Then $|y| = 1$ a.e. on S^1.

Since the map

$$u \mapsto \int_{\mathbb{D}_1} \big(|\nabla u|^2 - 2\,\mathrm{Jac}\, u\big)$$

is convex and continuous for $u \in H^1(\mathbb{D}_1)$ (and, thus, weakly l.s.c.), Eq. (4.19) and the fact that $y_{k_n} \rightharpoonup y$ imply

$$\int_{\mathbb{D}_1} \big(|\nabla y|^2 - 2\,\mathrm{Jac}\, y\big) = 4 \int_{\mathbb{D}_1} |\partial_{\bar z} y|^2 \leqslant 0. \tag{4.20}$$

Thus $\partial_{\bar z} y = 0$ a.e. in $\mathbb{D}_1$, that is y is holomorphic in $\mathbb{D}_1$. Set $g = y|_{S^1} \in H^{1/2}(S^1; S^1)$, whose Fourier expansion is of the form $g = \sum_{l=0}^{\infty} a_l e^{\imath l\theta}$. Then $\deg g = \sum_{l=0}^{\infty} l|a_l|^2$ (when g is smooth, this equation is equivalent to the degree formula (1.3); the same equality still holds for a general $g \in H^{1/2}(S^1; S^1)$ [14]). On the other hand, since y is holomorphic, it is the harmonic extension of g, hence

$$\int_{\mathbb{D}} |\nabla y|^2 = 2\pi \sum_{l=0}^{\infty} l|a_l|^2 = 2\pi \deg g \leqslant 2\pi, \tag{4.21}$$

where the last inequality follows from (4.18). Therefore, either $\deg g = 0$ and y is a constant of modulus 1 or $\deg g = 1$.

First, we rule out the possibility that y is a constant. For a large k, the set

$$M_k := \big\{z \in \Omega;\ |z - \zeta_k| \leqslant C_1 d(\zeta_k, \partial\Omega) \text{ and } d(z, \partial\Omega) \geqslant C_2 d(\zeta_k, \partial\Omega)\big\}$$

is contained in U and thus $|\Delta v_1^k| = \lambda |v^k(1 - |v^k|^2)| \leqslant \lambda$ in M_k. Using Lemmas 13(ii) and 12, we find that

$$|\Delta y_k| = \frac{1}{2}\big|\nabla \Phi_k^{-1}\big|^2 \big|\big(\Delta v_1^k\big) \circ \Phi_k^{-1}\big| \to 0 \quad \text{uniformly in } \mathbb{D}_r \text{ as } k \to \infty. \tag{4.22}$$

Since (y_k) is bounded in H^1, it follows from standard elliptic estimates that (y_k) is relatively compact in $C^1_{\mathrm{loc}}(\mathbb{D}_1)$. In particular, $y_{k_n} \to y$ uniformly in $\mathbb{D}_{1/2}$. Recalling that $y_k(0) = 0$, we find that $y(0) = 0$, that is, y cannot be a constant of modulus 1.

Next, we identify y. Lemma 7 applied to $g_n \equiv g$ implies that $|y(z)| \to 1$ uniformly as $|z| \to 1$. We recall that a holomorphic map y in $\mathbb{D}$ satisfying $|y(z)| \to 1$ uniformly as $|z| \to 1$ is a Blaschke product, i.e.

$$y(z) = \alpha \prod_{j=1}^{d} \frac{z - a_j}{1 - \overline{a_j} z}$$

for some $\alpha \in S^1$ and $a_1, \ldots, a_d \in \mathbb{D}$ [21]. Here d is the degree of $y|_{S^1}$. In our case $d = 1$ and $y(0) = 0$, thus $y = \alpha\,\mathrm{id}$ with $\alpha \in S^1$. Since $\partial_z y_k(0) \geqslant 0$, we have $\alpha = \partial_z y(0) \geqslant 0$, hence $\alpha = 1$ and $y = \mathrm{id}$.

The uniqueness of the weak limit implies that $y_k \rightharpoonup \mathrm{id}$ in H^1. Formula (4.18) combined with the fact that $\int_{\mathbb{D}} |\nabla\,\mathrm{id}\,|^2 = 2\pi$ yields $y_k \to \mathrm{id}$ in H^1. Further, since the sequence (y_k) is relatively compact in $C^1_{\mathrm{loc}}(\mathbb{D})$, it follows that $y_k \to \mathrm{id}$ in $C^1_{\mathrm{loc}}(\mathbb{D})$. $\square$

Step 5. Holomorphic (anti-holomorphic) behavior of v^k near $\partial\Omega$ $(\partial\omega)$.

As an immediate consequence of Lemma 14, we obtain the following lemma.

Lemma 15. *We have $v^k - \Phi_k \to 0$ in $L^2_{\mathrm{loc}}(\overline{A} \setminus \partial\omega)$ and $v^k - \overline{\Psi}_k \to 0$ in $L^2_{\mathrm{loc}}(\overline{A} \setminus \partial\Omega)$.*

Proof. We prove the first assertion. Fix a compact $K \subset \overline{A} \setminus \partial\omega$. Since the curves γ and Γ introduced in Step 2 are arbitrary, we have, thanks to Lemma 11,

$$\int_{K\setminus U} \left|\nabla v^k\right|^2 \to 0 \quad \text{as } k \to \infty. \tag{4.23}$$

On the other hand, Lemma 13(i) and the fact that $d(\zeta_k, \partial\Omega) \to 0$ imply that $\Phi_k(K \setminus U) \subset \mathbb{D} \setminus \mathbb{D}_{r_k}$ for some sequence $r_k \to 1$. The conformal invariance of the Dirichlet integral yields

$$\int_{K\setminus U} |\nabla\Phi_k|^2 = \int_{\Phi_k(K\setminus U)} |\nabla\,\mathrm{id}\,|^2 \leqslant \int_{\mathbb{D}\setminus\mathbb{D}_{r_k}} |\nabla\,\mathrm{id}\,|^2 \to 0 \quad \text{as } k \to \infty. \tag{4.24}$$

Finally,

$$\int_{K\cap U} \left|\nabla\Phi_k - \nabla v^k\right|^2 \leqslant \int_{U} \left|\nabla\Phi_k - \nabla v^k\right|^2 = \int_{\Phi_k(U)} |\nabla\,\mathrm{id} - \nabla y_k|^2 \to 0 \quad \text{as } k \to \infty, \tag{4.25}$$

by Lemma 14 and the conformal invariance. The conclusion of Lemma 15 follows by combining the estimates (4.23)–(4.25). $\square$

Step 6. Uniqueness of zeros of v^k and their degrees of for large k.

We argue by contradiction and assume that, possibly up to a subsequence, v^k has two distinct zeros ζ_k and $\tilde{\zeta}_k$ in U. Without loss of generality, we may further assume that

$$d(\zeta_k, \partial\Omega) \geqslant d(\tilde{\zeta}_k, \partial\Omega). \tag{4.26}$$

Let Φ_k and $\widetilde{\Phi}_k$ be the corresponding conformal representations. Given any $r \in (0,1)$, we claim that $\Phi_k^{-1}(\mathbb{D}_r) \cap \widetilde{\Phi}_k^{-1}(\mathbb{D}_r) = \emptyset$ for a sufficiently large k. Indeed, suppose that $z \in \Phi_k^{-1}(\mathbb{D}_r) \cap \widetilde{\Phi}_k^{-1}(\mathbb{D}_r)$ and let C_1 be as defined in Lemma 13. We have

$$|z - \zeta_k| \leqslant C_1 d(\zeta_k, \partial\Omega), \qquad |z - \tilde{\zeta}_k| \leqslant C_1 d(\tilde{\zeta}_k, \partial\Omega), \tag{4.27}$$

by Lemma 13(i) and, therefore

$$|\tilde{\zeta}_k - \zeta_k| \leqslant 2C_1 d(\zeta_k, \partial\Omega). \tag{4.28}$$

Equations (4.26) and (4.28), along with Lemma 13(iii) imply the existence of some fixed $\rho \in (0,1)$ such that $\Phi_k(\tilde{\zeta}_k) \in \overline{\mathbb{D}}_\rho$ for every $k \in \mathbf{N}$. However, this is impossible for large k, since on the one hand $y_k = v^k \circ \Phi_k^{-1} \to \mathrm{id}$ in $C^1(\overline{\mathbb{D}}_\rho)$ (and thus, for large k, $y_k|_{\overline{\mathbb{D}}_r}$ is into), while on the other hand $y_k(\Phi_k(\zeta_k)) = y_k(\Phi_k(\tilde{\zeta}_k)) = 0$ for each k. The claim is proved.

Now fix $r \in (1/\sqrt{2}, 1)$ so that

$$\int_{\mathbb{D}_r} |\nabla \,\mathrm{id}\,|^2 = 2\pi r^2 > \pi.$$

Setting $\tilde{y}_k = v^k \circ \widetilde{\Phi_k^{-1}}$, we obtain from Lemma 14 that

$$\frac{1}{2}\int_U |\nabla v^k|^2 \geqslant \frac{1}{2}\int_{\Phi_k^{-1}(\mathbb{D}_r)\cup\tilde{\Phi}_k^{-1}(\mathbb{D}_r)} |\nabla v^k|^2 = \frac{1}{2}\int_{\mathbb{D}_r} |\nabla y_k|^2 + \frac{1}{2}\int_{\mathbb{D}_r} |\nabla \tilde{y}_k|^2 \to 2\pi r^2, \tag{4.29}$$

as $k \to \infty$. Given our choice of r, Eq. (4.29) contradicts Eq. (4.13) thus proving the uniqueness of ζ_k.

Next, we determine, for large k, the degree of v^k around ζ_k. Since $y_k \to \mathrm{id}$ strongly in C^1_{loc} and $y_k(0) = 0$, it follows for large k that y_k has a zero of degree 1 at the origin. Since the diffeomorphism Φ_k is orientation preserving, we find that v^k has a zero of degree 1 at ζ_k for large k. Similarly, v^k has a zero of degree -1 at ξ_k for large k. $\square$

Proof of Theorem 4: case (a). Our purpose is to describe the behavior of a family (u_λ) of minimizers of (1.1)–(1.2) as $\lambda \to \infty$. The proof follows essentially the same lines as the one in case (b). We point out the changes that have to be made.

Step 1 is not needed here, since the minimizers already satisfy the GL equation and the property $|u_\lambda| \leqslant 1$. The equations

$$\lambda \int_A \left(1 - |u_\lambda|^2\right)^2 \to 0, \tag{4.30}$$

$$\|\nabla u_\lambda\|_{L^\infty(W)} \to 0, \tag{4.31}$$

$$\|\partial_{\bar{z}} u_\lambda\|_{L^2(U)} \to 0 \quad \text{and} \quad \|\partial_z u_\lambda\|_{L^2(V)} \to 0, \tag{4.32}$$

$$\frac{1}{2}\int_U |\nabla u_\lambda|^2 \to \pi \quad \text{and} \quad \int_U \mathrm{Jac}\, u_\lambda \to \pi, \tag{4.33}$$

$$\frac{1}{2}\int_V |\nabla u_\lambda|^2 \to \pi \quad \text{and} \quad \int_V \mathrm{Jac}\, u_\lambda \to -\pi \tag{4.34}$$

correspond to (4.10)–(4.14) in Step 2. However, while (4.10)–(4.14) were obtained via (4.17), the estimate (3.12) has to be used in case (a). Note that, although we established (3.12) in the critical case, the only assumption that needed there was that all possible weak-H^1 limits of sequences (u_{λ_n}) are constants. Hence (3.12) is still valid in the present context.

Using the same proof as in Step 3 in case (b), we find for large λ that u_λ has zeros ζ_λ and ξ_λ in U and in V, respectively. Moreover,

Lemma 16. *We have that* $\lambda^{1/2}d(\zeta_\lambda,\partial\Omega)\to 0$ *and* $\lambda^{1/2}d(\xi_\lambda,\partial\omega)\to 0$ *as* $\lambda\to\infty$.

Proof. We establish the first assertion. By (3.7), we have for some constant C independent of large λ that

$$\left|\nabla u_\lambda(z)\right| \leqslant \frac{C}{d(\zeta_\lambda,\partial\Omega)} \quad \text{if } |z-\zeta_\lambda| \leqslant \frac{1}{2}d(\zeta_\lambda,\partial\Omega). \tag{4.35}$$

Thus, choosing $c_\lambda = \frac{1}{2}\operatorname{Min}\{1, 1/C\}\, d(\zeta_\lambda,\partial\Omega)$, we have $\mathbb{D}_{c_\lambda}(\zeta_\lambda)\subset A$ and $|u_\lambda|\leqslant 1/2$ in $\mathbb{D}_{c_\lambda}(\zeta_\lambda)$. Therefore,

$$\lambda\int\limits_A \left(1-|u_\lambda|^2\right)^2 \geqslant \lambda \int\limits_{\mathbb{D}_{c_\lambda}(\zeta_\lambda)} \left(1-|u_\lambda|^2\right)^2 \geqslant \frac{9\pi\lambda c_\lambda^2}{16}. \tag{4.36}$$

The conclusion of Lemma 16 follows by combining (4.30) with (4.36). □

Next, we consider the rescaled maps $y_\lambda = u_\lambda\circ\Phi_\lambda^{-1}$ and $z_\lambda = \overline{u_\lambda\circ\Psi_\lambda^{-1}}$, where Φ_λ and Ψ_λ are suitable conformal representations vanishing at ζ_λ and ξ_λ, respectively. Step 4 works using the same proof as before except when establishing the analog of (4.22), which is

$$|\Delta y_\lambda| \to 0 \quad \text{in } C^0_{\mathrm{loc}}(\mathbb{D}). \tag{4.37}$$

The argument that leads to (4.37) is as follows. Let $r\in(0,1)$ be given. By combining Lemma 13(i) and (ii) with Lemma 16, we have

$$\|\Delta y_\lambda\|_{L^\infty(\mathbb{D}_r)} = \frac{1}{2}\left\|\left|\nabla\Phi_\lambda^{-1}\right|^2(\Delta u_\lambda)\circ\Phi_k^{-1}\right\|_{L^\infty(\Phi_\lambda^{-1}(\mathbb{D}_r))} \leqslant C_3\lambda d^2(\zeta_\lambda,\partial\Omega)\to 0, \tag{4.38}$$

as $\lambda\to\infty$.

Finally, Steps 5 and 6 are the same, and no changes are needed in the proof. □

Acknowledgments

The authors thank H. Brezis for valuable discussions. They are also grateful to D. Golovaty for careful reading of the manuscript and useful suggestions. The work of L.B. was supported by NSF grant DMS-0204637. The work of P.M. is part of the RTN Program "Fronts-Singularities". This work was initiated while both authors were visiting Rutgers University; part of the work was done while L.B. was visiting Université Paris-Sud and P.M. was visiting Penn State University. They thank the Mathematics Departments at these universities for their hospitality.

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